Roots of characteristic polynomials and intersection points of line arrangements

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Abstract

We study a relation between roots of characteristic polynomials and intersection points of line arrangements. Using these results, we obtain a lot of applications for line arrangements. Namely, we give (i) a generalized addition theorem for line arrangements, (ii) a generalization of Faenzi-Vallès' theorem over a field of arbitrary characteristic, (iii) a partial result on the conjecture of Terao for line arrangements, and (iv) a new sufficient condition for freeness over finite fields.

1 Main result

We use the notation in section two to state the main results in this article. Here some basic and special notations will be explained, which will be defined again in the next section.

Let $\mathbb{K}$ be a field of arbitrary characteristic and consider affine line arrangements in $V = \mathbb{K}^2$. We say an affine line arrangement $\mathcal{A}$ is **free** with exponents $\exp_0(\mathcal{A}) = (d_1, d_2)$ if the cone $c\mathcal{A}$ of $\mathcal{A}$ is free with exponents $(1, d_1, d_2)$. For a line H, define $\mathcal{A} \cap H := \{H \cap H' \neq \emptyset \mid H' \in \mathcal{A},\ H' \neq H\}$. Namely, this is the set of intersection points on H. Put $n_H := |\mathcal{A} \cap H|$ and let $\chi(\mathcal{A}, t)$ be the characteristic polynomial of $\mathcal{A}$. Now let us state the main result in this article.

Theorem 1.1
Let $\mathcal{C}$ be an affine line arrangement and assume that $\chi(\mathcal{C}, t) = (t-a)(t-a-b)$ with $a, b \in \mathbb{C}$ and $|a| \le |a+b|$. Then
(1) there are no $H \in \mathcal{C}$ such that $|a| < |\mathcal{C} \cap H| < |a+b|$. In other words, $\chi(\mathcal{C}, n_H) \ge 0$.
(2) There are no line $L \notin \mathcal{C}$ such that $|a| < |\mathcal{C} \cap L| < |a+b|$. In other words,

$\chi(\mathcal{C}, n_L) \geq 0$.
(3) Assume that $a, b \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{C}$ is free if there is a line H such that $|\mathcal{C} \cap H| = a$ or $a+b$. Equivalently, $\mathcal{C}$ is free if $\chi(\mathcal{C}, n_H) = 0$ for some line H.

If we assume the freeness, then we can obtain a stronger geometric condition on the arrangement.

Corollary 1.2
In the same notation as in Theorem 1.1, assume that $\mathcal{C}$ is free. Then
(1) $|\mathcal{C} \cap H| \in \mathbb{Z}_{\leq a} \cup \{a+b\}$ for any $H \in \mathcal{C}$, and
(2) $|\mathcal{C} \cap L| \in \{a\} \cup \mathbb{Z}_{\geq a+b}$ for any line $L \notin \mathcal{C}$.

Remark 1.3
(1) Theorem 1.1 (1) and (2) are non-trivial statements only when $a, b \in \mathbb{R}$ and $a < a+b$.
(2) Theorem 1.1 (1) gives some restriction on $H \in \mathcal{C}$ in terms of roots of $\chi(\mathcal{C}, t)$. On the other hand, Theorem 1.1 (2) seems to be more interesting. That is, the roots give a restriction on lines which are not belonging to $\mathcal{C}$. Hence Theorem 1.1 (2) says that combinatorics of $\mathcal{C}$ knows some information on geometry of $\mathcal{C}$.
(3) The case $n_H = a+b$ of Theorem 1.1 (3) when $b > 0$ is already known to experts. See [WY] for example.

Let us check the statement in Theorem 1.1 and Corollary 1.2 in the following example.

Example 1.4
(1) The simplest but important example is a set of n-lines $\mathcal{A}$ in the real plane which go through the origin. Then it is obvious that $n_H = 1$ for $H \in \mathcal{A}$, $n_L \in \{1, n-1, n\}$ for a line $L \notin \mathcal{A}$ and $\chi(\mathcal{A}, t) = (t-1)(t-n+1)$. This is trivial by using the property of parallel lines, but Theorem 1.1 says that this holds true for all line arrangements.
(2) Let $\mathcal{A}$ be an affine line arrangement in $\mathbb{R}^2$ defined by

$$x(x^2-y^2)(x^2-4y^2)(2x^2-y^2)(y-1) = 0.$$

Hence $|\mathcal{A}| = 8$ and

$$\chi(\mathcal{A}, t) = t^2 - 8t + 13 = (t-4-\sqrt{3})(t-4+\sqrt{3}).$$

Hence Theorem 1.1 (1) and (2) say that $|H \cap \mathcal{A}| \neq 3, 4, 5$. In fact, we can check that $|H \cap \mathcal{A}| \in \{2, 7\}$ for $H \in \mathcal{A}$ and $|\mathcal{A} \cap L| \in \{1, 2, 6, 7, 8\}$ for $L \notin \mathcal{A}$.
(3) Let $\mathcal{A}$ be an affine line arrangement in $\mathbb{R}^2$ defined by

$$xy(x^2-1)(y^2-1)(x^2-y^2)(x+y+1)(x+y-1)(x-y+1)(x-y-1) = 0.$$

Then $\chi(\mathcal{A},t)=(t-5)(t-7)$, and it is easy to check that $|\mathcal{A}\cap H|=3$ or 5 for any $H\in\mathcal{A}$, which matches Theorem 1.1 (1). Since we can check that there are no line $L\notin\mathcal{A}$ such that $|L\cap\mathcal{A}|=6$, Theorem 1.1 (2) is satisfied. Also, Theorem 1.1 (3) shows that $\mathcal{A}$ is free.

The proofs of Theorem 1.1 and Corollary 1.2 are simple, but we need algebraic methods for the proof of Theorem 1.1. In particular, recent developments on exponents of two-dimensional multiarrangements (e.g., [Yo], [WY] and [AN]) play the key roles.

Recall that the coefficients of $\chi(\mathcal{C},t)$ are the Betti numbers of the open manifold $V\setminus\cup_{H\in\mathcal{C}}H$ when $\mathbb{K}=\mathbb{C}$. Also, $\chi(\mathcal{C},t)$ can be computed combinatorially in the arrangement cases. Hence we are interested in topologocal and combinatorial proofs of them. As far as we investigated, there are no such results similar to them.

Also, these results have a lot of applications. For example, by using Theorem 1.1, we can generalize Faenzi-Vallès' theorem (Theorem 4.1) in [FV]. In Theorem 4.1, the key condition is the existence of a point with multiplicity n for the arrangement $\mathcal{A}$ with $\chi(\mathcal{A},t)=(t-n)(t-n-r)$. In this generalization, the role of this point is replaced by a free arrangement with exponents $(n-1,n-s)$ $(s\geq 1)$, i.e., the following holds.

Theorem 1.5
Let $\mathbb{K}$ be a field of arbitrary characteristic and $\mathcal{A}$ a line arrangement such that $|\mathcal{A}|=2n+r$ $(n,r\in\mathbb{Z}_{\geq 0})$ and $\chi(\mathcal{A},t)=(t-n)(t-n-r)$. Assume that $\mathcal{A}$ contains a free arrangement $\mathcal{B}$ with $\exp_0(\mathcal{B})=(n-s,n-1)$ $(s\geq 1)$. Then $\mathcal{A}$ is free if and only if there are no $\mathcal{B}\subset\mathcal{C}\subset\mathcal{A}$ such that $\chi(\mathcal{C},t)=(t-n-u+1)(t-n+s)$ with $u>r+1$.

If we remove the assumption that "$\mathcal{B}$ is free" from the statement in Theorem 1.5, then can we say something on freeness and combinatorics? In fact, we can also show the following combinatorial statement on freeness.

Theorem 1.6
Let $\mathbb{K}$ be a field of arbitrary characteristic and $\mathcal{A}$ a line arrangement such that $|\mathcal{A}|=2n+r$ $(n,r\in\mathbb{Z}_{\geq 0})$ and $\chi(\mathcal{A},t)=(t-n)(t-n-r)$. Assume that $\mathcal{A}$ contains an arrangement $\mathcal{B}$ with $\chi(\mathcal{B},t)=(t-\alpha)(t-\beta)$ such that $\alpha\leq\beta$ are real numbers with $\alpha\leq n$ and $n-1\leq\beta$. Then $\mathcal{A}$ is free if and only if there is a line $H\in\mathcal{A}$ such that $n_H\in\{n,n+r\}$. In particular, the freeness of such $\mathcal{A}$ depends only on combinatorics.

Another corollary is the following generalization of the addition theorem for line arrangements. To state it, let us introduce some terminologies. Define a **deletion pair of line arrangements** $(\mathcal{A},\mathcal{A}')$ by $\mathcal{A}\supset\mathcal{A}'$ and $|\mathcal{A}'|+1=$

$|\mathcal{A}|$. We say that a deletion pair $(\mathcal{A}, \mathcal{A}')$ is **free** if both $\mathcal{A}$ and $\mathcal{A}'$ are free. Then the following addition-type theorem holds.

Corollary 1.7
A deletion pair $(\mathcal{A}, \mathcal{A}')$ is free if and only if $\chi(\mathcal{A}, t)$ and $\chi(\mathcal{A}', t)$ have a common root. In particular, the freeness of the deletion pair depends only on the combinatorics.

Also, we apply Theorem 1.1 and Corollary 1.2 to obtain some results related to the conjecture of Terao (§5, Corollary 5.5) and free arrangements over finite fields (§6, Theorem 6.3).

The organization of this article is as follows. In §2 we introduce several definitions and results for the proof. In §3 we prove main theorems. In §4 we show generalized Faenzi-Vallès' theorem as Theorems 1.5 and 1.6. In §5 we show an application to the conjecture of Terao when one of the roots of the characteristic polynomial is at most five. In §6 we consider the case when the base field is a finite field.

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2 Preliminaries

In this section let us introduce several definitions and results, some of which have already defined in section one. We will use them throughout this article. We use [OT] as a general reference on arrangement theory. Also, a recent paper [Yo3] is a nice reference on exponents of two-dimensional multiarrangements.

Let $\mathbb{K}$ be a field of arbitrary characteristic unless otherwise specified, $V = \mathbb{K}^2$ and $S' = \mathrm{Sym}^*(V^*) \simeq \mathbb{K}[x, y]$ the coordinate ring of V. An **affine arrangement** $\mathcal{C}$ **of lines** in V is a finite collection of affine lines in V. Let $L(\mathcal{C}) := \{\cap_{H \in \mathcal{B}} H \neq \emptyset \mid \mathcal{B} \subset \mathcal{C}\}$ be the **intersection lattice** of $\mathcal{C}$. Define $\mu : L(\mathcal{C}) \to \mathbb{Z}$ by $\mu(V) = 1$, and by $\mu(X) := -\sum_{X \subsetneq Y \subset V} \mu(Y)$. Then the **characteristic polynomial** $\chi(\mathcal{C}, t)$ of $\mathcal{C}$ is defined by

$$\chi(\mathcal{C}, t) := \sum_{X \in L(\mathcal{C})} \mu(X) t^{\dim X} = t^2 - |\mathcal{C}|t + b_2(\mathcal{C}).$$

For a line H, define $H \cap \mathcal{C} := \{H \cap H' \neq \emptyset \mid H' \in \mathcal{C},\ H' \neq H\}$ and put $n_H := |H \cap \mathcal{C}|$.

Let z be a new coordinate and define the **cone** $c\mathcal{C}$ of C as follows. If $\mathcal{C}$ is defined by a non-homogeneous polynomial equation $Q = 0$, then $c\mathcal{C}$ is

defined by $z(cQ) = 0$, where cQ is the homogenized polynomial of Q by the coordinate z. Hence $c\mathcal{C}$ is a **central** arrangement in $\mathbb{K}^3$, i.e., all planes contain the origin. For $H \in \mathcal{C}$, let $cH \in c\mathcal{C}$ denote the homogenized linear plane of H. Let $S := \mathbb{K}[x, y, z]$ and $\mathrm{Der}\, S$ be the module of S-derivations with a basis $\partial_x, \partial_y, \partial_z$ dual to x, y, z respectively. Let α_{cH} be a defining linear form of $cH \in c\mathcal{C}$. Hence the defining polynomial $Q(c\mathcal{C})$ of the cone $c\mathcal{C}$ of $\mathcal{C}$ is $z(\prod_{H\in\mathcal{C}} \alpha_{cH})$. Then define

$$\begin{aligned} D(c\mathcal{C}) : &= \{\theta \in \mathrm{Der}\, S \mid \theta(\alpha_{cH}) \in S\alpha_{cH}\ (\forall H \in \mathcal{C}),\ \theta(z) \in Sz\}, \\ D_0(c\mathcal{C}) : &= \{\theta \in D(c\mathcal{C}) \mid \theta(z) = 0\}. \end{aligned}$$

We say that $c\mathcal{C}$ is **free** with **exponents** $\exp(c\mathcal{C}) = (1, d_1, d_2)$ if $D(c\mathcal{C})$ is a free S-module with homogeneous basis elements $\theta_E = x\partial_x + y\partial_y + z\partial_z,\ \theta_1,\ \theta_2$ such that $\deg \theta_i = d_i$. We say that an affine arrangement $\mathcal{C}$ is free with exponents $\exp_0(\mathcal{C}) = (d_1, d_2)$ if $c\mathcal{C}$ is free with $\exp(c\mathcal{C}) = (1, d_1, d_2)$.

Let $\mathcal{A}$ be a central line arrangement and $m : \mathcal{A} \to \mathbb{Z}_{>0}$ be a **multiplicity**. Here α_H denotes a defining linear form of $H \in \mathcal{A}$. Then a pair $(\mathcal{A}, m)$ is called a **multiarrangement** and we can define the logarithmic module

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}\, S' \mid \theta(\alpha_H) \in S'\alpha_H^{m(H)}\ (\forall H \in \mathcal{A})\}.$$

Let $Q(\mathcal{A}, m) := \prod_{H\in\mathcal{A}} \alpha_H^{m(H)}$. Since S' is two-dimensional, $D(\mathcal{A}, m)$ is always free. Hence we can always define its exponents $\exp(\mathcal{A}, m) := (d_1, d_2)$. Here we introduce a very famous freeness criterion.

Theorem 2.1 (Saito's criterion, [Sa], [Zi])
Let $\theta_1, \theta_2 \in D(\mathcal{A}, m)$ be two derivations with $\deg \theta_i = d_i$. Then they form a basis for $D(\mathcal{A}, m)$ if and only if θ_1 and θ_2 are S'-independent and $d_1 + d_2 = |m| := \sum_{H\in\mathcal{A}} m(H)$.

For an affine line arrangement $\mathcal{C}$ and $H_0 \in c\mathcal{C}$, let $(\mathcal{C}'', m)$ be the **Ziegler restriction** of $c\mathcal{C}$ onto H_0 defined by $C'' := \{H \cap H_0 \mid H \in c\mathcal{C} \setminus \{H_0\}\}$ and by

$$m(H \cap H_0) := |\{H' \in c\mathcal{C} \setminus \{H_0\} \mid H' \cap H_0 = H \cap H_0\}|.$$

The **Ziegler restriction of $\mathcal{C}$ onto** $H \in \mathcal{C}$ is that of $c\mathcal{C}$ onto $cH \in c\mathcal{C}$. Let $(d_1(\mathcal{C}), d_2(\mathcal{C}))$ denote the exponents of the Ziegler restriction of $c\mathcal{C}$ onto $z = 0$. In general, we assume that $d_1(\mathcal{C}) \le d_2(\mathcal{C})$. Then the following is the key theorem in this article.

Theorem 2.2 ([Yo], Theorem 3.2)
Let $\exp(\mathcal{C}'', m) = (d_1, d_2)$. Then $\chi(\mathcal{C}, 0) = b_2(\mathcal{C}) \ge d_1 d_2$, and the equality holds if and only if $\mathcal{C}$ is free.

Also, we use the results in the following papers; [T], [T1], [Zi], [WY], [AN] and [A2]. For the proof and application of main results, let us introduce some of them.

First, let us introduce three results. Namely, the first one is the addition theorem in [T], the second the factorization theorem in [T1], and the third the Ziegler's restriction theorem in [Zi]. Note that all of these three were proved for any dimensional arrangements in these papers. Since we focus on line arrangements, we introduce the line arrangement cases of these results as follows.

Theorem 2.3 (Addition theorem, [T])
Let $\mathcal{A}$ be an affine line arrangement and fix $H \in \mathcal{A}$. Define $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $n_H := |\mathcal{A} \cap H|$. Assume that $\chi(\mathcal{A}, n_H) = \chi(\mathcal{A}', n_H) = 0$. Then $\mathcal{A}$ is free if and only if $\mathcal{A}'$ is free.

Theorem 2.4 (Factorization theorem, [T1])
Let $\mathcal{A}$ be a free affine line arrangement with $\exp_0(\mathcal{A}) = (d_1, d_2)$. Then $\chi(\mathcal{A}, t) = (t - d_1)(t - d_2)$.

Theorem 2.5 ([Zi])
If $\mathcal{A}$ is a free affine line arrangement with $\exp_0(\mathcal{A}) = (a, b)$, then its Ziegler restriction $(\mathcal{A}'', m)$ is free with $\exp(\mathcal{A}'', m) = (a, b)$.

The statements of Corollary 1.7 and Theorem 2.3 are similar, and it is easy to see that the former is a generalization of the latter. The next two results are originally for line arrangements. The first one is originally in [WY].

Lemma 2.6 ([AN], Lemma 4.2, Lemma 4.3)
Let $\mathcal{A}$ be a central line arrangement and m, m' be multiplicities on $\mathcal{A}$ such that $|m| = |m'| + 1$ and $m(H) \geq m'(H)$ for any $H \in \mathcal{A}$. If $\exp(\mathcal{A}, m') = (d_1, d_2)$, then $\exp(\mathcal{A}, m) = (d_1 + 1, d_2)$ or $(d_1, d_2 + 1)$.

Theorem 2.7 ([A2])
Let $\mathcal{A}$ be an affine line arrangement defined over a field of characteristic zero. Put $\chi(\mathcal{A}, t) = (t - \alpha)(t - \beta)$ for $\alpha, \beta \in \mathbb{C}$. For the Ziegler restriction $(\mathcal{A}'', m)$ of $\mathcal{A}$ onto $H_0 \in \mathcal{A}$, put $\exp(\mathcal{A}'', m) = (d_1, d_2)$ with $d_1 \leq d_2$. Assume that $|m| \geq 2m(H)$ for any $H \in \mathcal{A}''$ and $|\mathcal{A}''| =: h > 2$. Then
(1) $d_2 - d_1 \leq h - 2$, and
(2) $||\alpha| - |\beta|| \leq h - 2$. In particular, $\mathcal{A}$ is free if $||\alpha| - |\beta|| \in \{h - 2, h - 3\}$.

Proof. The statement (1) is the same as Theorem 3.5 in [A2]. Also, the statement (2) is essentially proved in [A2]. That is, combine $\mathbb{Z} \ni \alpha\beta \geq d_1 d_2$

(by Theorem 2.2) and $\alpha + \beta = d_1 + d_2 = |\mathcal{A}| = |m|$ with (1) and Theorem 2.2. $\square$

When $(\mathcal{A}'', m)$ satisfies the condition $|m| \geq 2m(H)$ for any $H \in \mathcal{A}''$ in Theorem 2.7, we say that $(\mathcal{A}'', m)$, or $\mathcal{A}$ is **balanced**. The following is famous in the theory of two-dimensional multiarrangements. We give a proof for the completeness.

Lemma 2.8
Let $\mathcal{A}$ be an affine line arrangement which is not balanced. Then the freeness of $\mathcal{A}$ depends only on $L(\mathcal{A})$.

Proof. By definition, the Ziegler restriction $(\mathcal{A}'', m)$ of $\mathcal{A}$ is not balanced. We may assume that $H_0 := \{x = 0\}$ satisfies $2m(H_0) > |m|$. Let $\varphi := (Q(\mathcal{A}'', m)/x^{m(H_0)})\partial_y$. Then clearly $\varphi \in D(\mathcal{A}'', m)$ is a non-zero derivation of the smallest degree. Hence $\exp(\mathcal{A}'', m)$ is combinatorially determined as $(|m| - m(H_0), m(H_0))$ and Theorem 2.1 completes the proof. $\square$

Some of the following statements are well-known (e.g., see [WY]), but we give the proof for the completeness.

Lemma 2.9
Let $\mathcal{A}$ be a central line arrangement with $|\mathcal{A}| = n$, m be a multiplicity on $\mathcal{A}$ and put $\exp(\mathcal{A}, m) = (d_1, d_2)$ with $d_1 \leq d_2$.
(1) If $|m| \geq 2n - 2$, then $d_i \geq n - 1$.
(2) If $|m| \leq 2n - 2$, then $d_1 = |m| - n + 1$, $d_2 = n - 1$.
(3) Let $|m| = \alpha + \beta$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. If $\alpha < n - 1 < \beta$, then $\alpha < d_1 \leq d_2 < \beta$.

Proof. (1) Note that $\exp(\mathcal{A}) = (1, n - 1)$. Take any multiplicity m' such that $m(H) \geq m'(H) \geq 1$ for any $H \in \mathcal{A}$ and $|m'| = 2n - 2$. Let θ_E be the Euler derivation. Then it is easily checked that $\theta := (Q(\mathcal{A}, m')/Q(\mathcal{A}))\theta_E \in D(\mathcal{A}, m')$ is a non-zero element in $D(\mathcal{A}', m)$ of degree $n-1$ such that there are no $\theta' \in \operatorname{Der} S'$ satisfying $f\theta' = \theta$ for $f \in S'$ with $\deg f > 0$. Hence Theorem 2.1 implies that $\exp(\mathcal{A}, m') = (n - 1, n - 1)$. Since $D(\mathcal{A}, m') \supset D(\mathcal{A}, m)$, we complete the proof.
(2) Use the same $\theta = (Q(\mathcal{A}, m)/Q(\mathcal{A}))\theta_E$ as in the proof of (1). Then $\deg \theta = |m| - n + 1$ and it is clear that θ is a non-zero element of $D(\mathcal{A}, m)$ of the smallest degree. Hence Theorem 2.1 completes the proof.
(3) First assume that $\alpha \geq d_1$. Then the construction of θ in the proofs above shows that $d_2 = n - 1$. Hence $|m| = d_1 + d_2 \leq \alpha + n - 1 < \alpha + \beta = |m|$, which is a contradiction. Hence $d_1 > \alpha$. Assume that $d_2 \geq \beta$. Then $|m| = d_1 + d_2 > \alpha + \beta = |m|$, which is a contradiction. Hence $d_2 < \beta$. $\square$

Lemma 2.10
Let $\mathcal{A}$ be a central line arrangement and m, m' be multiplicities on $\mathcal{A}$ such that $m(H) \geq m'(H)$ for any $H \in \mathcal{A}$. Put $\exp(\mathcal{A}, m') = (d_1, d_2)$ and $\exp(\mathcal{A}, m) = (e_1, e_2)$ with $d_1 \leq d_2$, $e_1 \leq e_2$. Then $d_1 \leq e_1$, $d_2 \leq e_2$.

Proof. Let θ_1, θ_2 (resp:φ_1, φ_2) be a basis for $D(\mathcal{A}, m')$ (resp:$D(\mathcal{A}, m)$) with $\deg \theta_i = d_i$ (resp:$\deg \varphi_i = e_i$). Since $D(\mathcal{A}, m) \subset D(\mathcal{A}, m')$, it is clear that $e_1 \geq d_1$. Assume that $e_2 < d_2$. Then $\varphi_2 = f\theta_1$ for $f \in S'$. Put $\varphi_1 = g\theta_1 + h\theta_2$ for $g, h \in S'$. Then the inequality $e_1 \leq e_2 < d_2$ shows that $h = 0$. Hence φ_1 and φ_2 are S'-dependent, which is a contradiction. $\square$

The next proposition may be known to experts, but we give a proof for the completeness.

Proposition 2.11
Let $\mathcal{A} \supset \mathcal{B}$ be affine line arrangements such that $\chi(\mathcal{A}, t) = (t-a)(t-c)$, $\chi(\mathcal{B}, t) = (t-a)(t-b)$ with $a, b, c \in \mathbb{Z}_{\geq 0}$. Assume that $a \leq b \leq c$. Then $\mathcal{A}$ is free if $\mathcal{B}$ is free.

Proof. Assume that $\mathcal{B}$ is free. Then $(d_1(\mathcal{B}), d_2(\mathcal{B})) = (a, b)$ by Theorem 2.5. By Theorem 2.2, it suffices to show that $(d_1(\mathcal{A}), d_2(\mathcal{A})) = (a, c)$. If not, then Lemma 2.10 and Theorem 2.2 show a contradiction. $\square$

Example 2.12
The conditions in Proposition 2.11 are essential. Consider

$$\begin{aligned}
\mathcal{A} : &= xy(y^2-1)(x^2-4y^2)(x^2-9y^2), \\
\mathcal{B} : &= x(y-1)(x^2-4y^2)(x^2-9y^2), \\
\mathcal{C} : &= (x^2-4y^2)(x^2-9y^2).
\end{aligned}$$

Then $\exp_0(\mathcal{A}) = (3, 5)$, $\exp_0(\mathcal{C}) = (1, 3)$ and $\chi(\mathcal{B}, t) = (t-3)^2$, but $\mathcal{B}$ is not free.

3 Proof of Theorem 1.1 and Corollary 1.2

In this section we prove main results introduced in section one.

Proof of Theorem 1.1. If both a and b are not real numbers, then $|a| = |a+b|$. Hence there is nothing to prove. So in the proof below, we may assume that a and b are both real numbers. Also, we may assume that a and $a+b$ are both non-negative since the roots of $\chi(\mathcal{C}, t) = t^2 - |\mathcal{C}|t + b_2(\mathcal{C})$ are

apparently non-negative. Hence in the below, we may replace $|a|$ and $|a+b|$ by a and $a+b$ respectively.

(1) Assume that such $H \in \mathcal{C}$ exists. Let $(\mathcal{C}'', m)$ be the Ziegler restriction of $\mathcal{C}$ onto H and let $|C''| = n_H + 1$. Then $\exp(\mathcal{C}'') = (1, n_H)$ with $a < n_H < a+b$. Let $\exp(\mathcal{C}'', m) = (d_1, d_2)$ with $d_1 \le d_2$. Then it follows that $a < d_1 \le d_2 < a+b$ by Lemma 2.9 (3). Hence $d_1 d_2 > a(a+b) = b_2(\mathcal{C})$, which contradicts Theorem 2.2.

(2) If $b = 0$, then there is nothing to show. Note that the solutions a and $a+b$ of $\chi(\mathcal{C}, t) = 0$ are of the form

$$\frac{|\mathcal{C}| \pm \sqrt{|\mathcal{C}|^2 - 4b_2(\mathcal{C})}}{2}.$$

Since $|\mathcal{C}|$ and $b_2(\mathcal{C})$ are both non-negative integers, $b = \sqrt{|\mathcal{C}|^2 - 4b_2(\mathcal{C})} \neq 0$ implies that $b \ge 1$.

Assume that there is a line $L \notin \mathcal{C}$ such that $a < n_L < a+b$. Let $\mathcal{A} := \mathcal{C} \cup \{L\}$. Since $\chi(\mathcal{A}, t) = t^2 - (2a+b+1)t + a(a+b) + n_L$, the solutions of $\chi(\mathcal{A}, t) = 0$ are

$$\frac{2a+b+1 \pm \sqrt{(b+1)^2 + 4(a - n_L)}}{2}.$$

Let $\alpha := (2a+b+1)/2$ and $\beta := (\sqrt{(b+1)^2 + 4(a-n_L)}\ )/2$. Note that $\beta \in \mathbb{R}$ since $(b+1)^2 \ge 4b$ and $a+b > n_L$. Since negative real numbers cannot be a solution of $\chi(\mathcal{A}, t)$, we have $\alpha \pm \beta \in \mathbb{R}_{\ge 0}$. Also, using $a < n_L < a+b$, easy computations show that

$$a < \alpha - \beta < a+1, \ a+b < \alpha + \beta < a+b+1.$$

Now apply Theorem 1.1 (1) to the arrangement $\mathcal{A}$ and $L \in \mathcal{A}$. Then we know that $a+1 \le n_L < a+b$ cannot occur. Hence to complete the proof, it suffices to show that $a < n_L < a+1$ cannot occur.

Assume that there exists the real number e such that $0 < e < 1$ and $n_L = a+e \in \mathbb{Z}$. Hence $a+e$ is the integer satisfying $a < a+e < a+1$. Then

$$\chi(\mathcal{A}, t) = t^2 - (2a+b+1)t + a(a+b) + a + e.$$

Let $(\mathcal{A}'', m)$ be the Ziegler restriction of $\mathcal{A}$ onto L. Put $\exp(\mathcal{A}'', m) = (d_1, d_2)$ with $d_1 \le d_2$. Assume that $d_1 \le a$. Then Lemma 2.9 (2) and the fact that $\exp(\mathcal{A}'') = (1, a+e)$ show that $d_2 = a+e$. Thus $2a+b+1 = d_1 + d_2 \le 2a + e < 2a+b+1$, which is a contradiction. Hence $a < d_1$. In particular,

$a+e \leq d_1$. Then Lemma 2.6 shows that $d_1 d_2 \geq (a+e)(a+b+1-e)$. So the inequalities $b \geq 1$ and $0 < e < 1$ imply that

$$\begin{aligned} b_2(\mathcal{A}) - d_1 d_2 &\leq a(a+b) + a + e - (a+e)(a+b+1-e) \\ &= a(a+b) - (a+e)(a+b-e) < 0, \end{aligned}$$

which contradicts Theorem 2.2.

(3) If $H \notin \mathcal{C}$, replace $\mathcal{C}$ by $\mathcal{C} \cup \{H\}$ and we may assume that $H \in \mathcal{C}$ by Theorem 2.3. First assume that $|\mathcal{C} \cap H| = a+b$. Let $(\mathcal{C}'', m)$ be the Ziegler restriction of $\mathcal{C}$ onto H. Then $\exp(\mathcal{C}'', m)$ is combinatorially determined as $(a, a+b)$ by Lemma 2.9 (2). Hence $\mathcal{C}$ is free by Theorem 2.2. Next assume that $|\mathcal{C} \cap H| = a$. Then $\exp(\mathcal{C}'') = (1, a)$. Hence Lemma 2.9 (1) shows that $d_i \geq a$ for $\exp(\mathcal{C}'', m) = (d_1, d_2)$. Again by Theorem 2.2, we know that $a(a+b) \geq d_1 d_2$. So Lemma 2.10 implies that $d_1 = a$, $d_2 = a+b$, which implies the freeness of $\mathcal{C}$ by Theorem 2.2. □

Proof of Corollary 1.2. (1) By Theorem 1.1, it suffices to show that $|\mathcal{C} \cap H| \leq a+b$ for $H \in \mathcal{C}$. Assume not. Then Lemma 2.9 (2) shows that $d_1 d_2 < a(a+b)$ for $\exp(\mathcal{C}'', m) = (d_1, d_2)$, which contradicts Theorem 2.2.

(2) First assume that $a = 0$. This occurs only when all lines in $\mathcal{C}$ are parallel. In this case, Corollary 1.2 is obvious. Hence we may assume that $a > 0$.

Since there is at least one point in $L(\mathcal{C})$ by the previous paragraph, it holds that $\chi(\mathcal{C}, 0) > 0$ and $|\mathcal{C}| \geq 2$. Also, it is well-known that $\chi(\mathcal{C}, 1) \geq 0$ (e.g., by Zaslavsky's theorem, [Za]). Since $1 \leq |\mathcal{C}|/2$, the non-negativity of $\chi(\mathcal{C}, 0)$ and $\chi(\mathcal{C}, 1)$ implies that $a \geq 1$. Hence in the arguments below, we assume that $a \geq 1$.

By Theorem 1.1, it suffices to show that $|\mathcal{C} \cap L| \geq a$ for any line $L \notin \mathcal{C}$. Assume not and put $C_1 := \mathcal{C} \cup \{L\}$. Let (C_1'', m_1) be the Ziegler restriction of $\mathcal{C}_1$ onto L and $n := |C_1 \cap L| < a$. Then $b_2(\mathcal{C}_1) = b_2(\mathcal{C}) + n$. On the other hand, $\exp(\mathcal{C}_1'', m_1) = (a+1, a+b)$ or $(a, a+b+1)$ because $\exp(\mathcal{C}'', m) = (d_1, d_2) = \exp_0(\mathcal{C}) = (a, a+b)$ and Lemma 2.6. Hence $a \geq 1$ implies that $b_2(\mathcal{C}_1) = b_2(C) + n < a(a+b+1) \leq (a+1)(a+b)$, which contradicts Theorem 2.2. □

Proof of Corollary 1.7. Let $a \in \mathbb{C}$ be a common root of $\chi(\mathcal{A}, t)$ and $\chi(\mathcal{A}', t)$. Then the famous deletion-restriction formula (e.g., see [OT], Corollary 2.57) shows that

$$0 = \chi(\mathcal{A}, a) = \chi(\mathcal{A}', a) - \chi(\mathcal{A} \cap H, a) = -\chi(\mathcal{A} \cap H, a),$$

where $\{H\} = \mathcal{A} \setminus \mathcal{A}'$. By definition, $\chi(\mathcal{A} \cap H, a) = a - n_H$. Hence $a = n_H \in$

$\mathbb{Z}_{\geq 0}$, and both characteristic polynomials factorize into

$$\begin{aligned}\chi(\mathcal{A}',t) &= (t-a)(t-b),\\ \chi(\mathcal{A},t) &= (t-a)(t-b-1).\end{aligned}$$

Thus Theorem 1.1 (3) shows the freeness of both arrangements. □

Remark 3.1
Corollary 1.7 makes several proofs of the freeness of line arrangements easier, especially those related to extended Catalan and Shi arrangements. For example in [A1], the freeness of several deformations of the Coxeter arrangements of the type A_2 are proved by checking all the addition steps. However, if we use Corollary 1.7, it suffices to find a line H on each deformations such that n_H is one of the roots of their characteristic polynomials.

Example 3.2
Theorem 1.1 (3) and Corollary 1.7 are useful as we saw above, but they are not enough to show freeness of all arrangements combinatorially. Recall the affine line arrangement $\mathcal{A}$ consisting of all edges and diagonals of a regular pentagon. Then $\chi(\mathcal{A},t) = (t-5)^2$ and $\mathcal{A}$ is free, but $|\mathcal{A} \cap H| = 4$ for any $H \in \mathcal{A}$. Hence we cannot apply Theorem 1.1 (3) and Corollary 1.7 to show its freeness combinatorially. Of course, it is easy to see that there is a line $L \notin \mathcal{A}$ such that $|\mathcal{A} \cap L| = 5$. Hence Theorem 1.1 (3) shows that $\mathcal{A}$ is free, but this proof is not combinatorial. Also, it is easy to check that $\mathcal{A}$ contains a free arrangement with exponents $(3,3)$, but $\mathcal{A}$ does not satisfy the sufficient condition of freeness in Theorem 1.5. Hence the condition in Theorem 1.5 is essential.

4 Proof of Theorems 1.5 and 1.6

Before the proof of Theorem 1.5 as an application of Theorem 1.1, let us recall the following Faenzi-Vallès' theorem.

Theorem 4.1 ([FV], Theorem 2)
Let $\mathcal{A}$ be an affine 2-arrangement in $V = \mathbb{C}^2$ such that $|\mathcal{A}| = 2n + r$ $(n, r \in \mathbb{Z}_{\geq 0})$ and that its localization $\mathcal{B} \subset \mathcal{A}$ at the origin consists of h-lines with $n \leq h \leq n + r + 1$. If $\chi(\mathcal{A},t) = (t-n)(t-n-r)$, then $\mathcal{A}$ is free.

If $\mathcal{A}$ contains a point with multiplicity h with $n \leq h \leq n + r + 1$, then it implies that $\mathcal{A}$ contains a free arrangement $\mathcal{B}$ with $\exp_0(\mathcal{B}) = (1, h-1)$ and $n - 1 \leq h - 1 \leq n + r$. Hence Theorem 1.5 generalizes Theorem 4.1 in

the sense of freeness. Also, note that Theorem 1.5 holds true not only over $\mathbb{C}$ but also all fields of any characteristic.

For the proof of Theorem 1.5, let us introduce the following corollary and lemma by using the results in the previous section. The first corollary might be similar to non-freeness criterion in [K].

Corollary 4.2
Let $\mathcal{A} \supset \mathcal{B}$ be an affine line arrangement such that $\chi(\mathcal{A}, t) = (t-a)(t-b)$, $\chi(\mathcal{B}, t) = (t-c)(t-d)$ with integers $a \le b$, $c \le d$ and $\mathcal{B}$ is free. If $|\mathcal{A} \cap H| < b$, then for $H \in \mathcal{A} \backslash \mathcal{B}$, it holds that $|\mathcal{B} \cap H| \in \{c\} \cup \{d, d+1, \ldots, a\}$.

Proof. Obvious by Theorem 1.1 and Corollary 1.2. $\square$

Lemma 4.3
Let $\mathcal{A}$ and $\mathcal{B}$ be affine line arrangements such that $\mathcal{A} \supset \mathcal{B}$ with $|\mathcal{A} \setminus \mathcal{B}| = f$. Then we can order lines of $\mathcal{A} \setminus \mathcal{B} = \{H_1, \ldots, H_f\}$ in such a way that, for $\mathcal{B}_0 := \mathcal{B}$, $\mathcal{B}_i := \mathcal{B}_{i-1} \cup \{H_i\}$ and $n_i := |\mathcal{B}_{i-1} \cap H_i|$, it holds that $n_1 \le n_2 \le \cdots \le n_f$.

Proof. We use induction on i. First, let $H_1 \in \mathcal{A} \setminus \mathcal{B}$ be a line such that $|\mathcal{B} \cap H_1| = \min_{H \in \mathcal{A} \backslash \mathcal{B}} |\mathcal{B} \cap H|$. Then for any $H \in \mathcal{A} \setminus (\mathcal{B} \cup \{H_1\})$, it is obvious that $|\mathcal{B} \cap H_1| \le |(\mathcal{B} \cup H_1) \cap H|$. Assume that $H_1, \ldots, H_i \in \mathcal{A} \setminus \mathcal{B}$ satisfy the condition in the statement. Then choose $H_{i+1} \in \mathcal{A} \setminus \mathcal{B}_i$ such that $|\mathcal{B}_i \cap H_{i+1}| = \min_{H \in \mathcal{A} \backslash \mathcal{B}_i} |\mathcal{B}_i \cap H|$. Then it is obvious that $n_i \le |\mathcal{B}_i \cap H|$ for any $H \in \mathcal{A} \setminus \mathcal{B}_i$. $\square$

Proof of Theorem 1.5. First assume that such $\mathcal{C}$ does not exist. If there is a line $H \in \mathcal{A}$ such that $n_H = n$ or $n + r$, then Theorem 1.1 shows that $\mathcal{A}$ is free. Assume that $n_H \ne n,\ n + r$. Again by Theorem 1.1, we may assume that $n_H < n$ or $n_H > n + r$. Also, by Corollaries 1.2 and 4.2, $|H \cap \mathcal{B}| \in \mathbb{Z}_{\ge n-1} \cup \{n - s\}$ for $H \in \mathcal{A} \setminus \mathcal{B}$.

Let $\mathcal{A} \setminus \mathcal{B} = \{H_1, \ldots, H_{r+s+1}\}$, $\mathcal{B}_0 := \mathcal{B}$ and $\mathcal{B}_i := \mathcal{B} \cup \{H_1\} \cup \cdots \cup \{H_i\}$. By Lemma 4.3, we may assume that $n_1 \le n_2 \le \cdots \le n_{r+s+1}$ for $n_i := |\mathcal{B}_{i-1} \cap H_i|$ $(i = 1, \ldots, r + s + 1)$. By the previous paragraph, we know that $\{n - s\} \cup \mathbb{Z}_{\ge n-1} \ni n_1 \le n_{r+s+1} \in \mathbb{Z}_{<n} \cup \mathbb{Z}_{>n+r}$. Note that $n_{r+s+1} = |\mathcal{A} \cap H_{r+s+1}|$.

Case 1. Assume that $n_1 = n - s$.

Case 1-1. Assume that $n_2 > n-s$. Then $\mathcal{B}_1$ is free with $\exp_0(\mathcal{B}_1) = (n, n-s)$. By Theorem 1.1, $n_2 \ge n$. Since $n \le n_2 \le n_{r+s+1} \ne n$, we have $n_{r+s+1} > n+r$ by Theorem 1.1. Hence

$$b_2(\mathcal{A}) > n(n-s) + (r+s-1)n + n + r = n(n+r) + r \ge n(n+r) = b_2(\mathcal{A}),$$

which is a contradiction.
Case 1-2. Assume that $n_1 = \cdots = n_u = n - s < n_{u+1}$ for some $u > 1$. Then $\mathcal{B}_u$ is free with $\exp_0(\mathcal{B}_u) = (n + u - 1, n - s)$. If $r \geq u - 1$, then $n_i \geq n + u - 1 > n - 1$ for $i > u$ by Corollary 1.2 and $n + u - 1 \leq n + r$. Hence

$$\begin{aligned} b_2(\mathcal{A}) &> (n + u - 1)(n - s) + (r + s + 1 - u)(n + u - 1) \\ &= (n + u - 1)(n + r + 1 - u) \geq n(n + r) = b_2(\mathcal{A}) \end{aligned}$$

because of $0 \leq r + 1 - u \leq r$ and $n_{r+s+1} > n + r$, which is a contradiction.

If $r < u - 1$, then there exists $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$ such that $\chi(\mathcal{C}, t) = (t - n - u + 1)(t - s)$ and $r < u - 1$, which contradicts the assumption.
Case 2. So we may assume that $n_1 \geq n - 1$. If $n_{r+s+1} = n - 1$, then

$$b_2(\mathcal{A}) = (n-1)(n-s)+(r+s+1)(n-1) = (n-1)(n+r+1) < n(n+r) = b_2(\mathcal{A}),$$

which is a contradiction. Hence $n_{r+s+1} \geq n$. By the assumption and Theorem 1.1, it holds that $n_{r+s+1} > n + r$. Hence

$$b_2(\mathcal{A}) > (n - 1)(n - s) + (r + s)(n - 1) + n + r = n(n + r) = b_2(\mathcal{A}),$$

which is a contradiction.

Conversely, assume that such $\mathcal{C}$ exists. Then Proposition 2.11 shows that $\mathcal{C}$ is also free. Hence its Ziegler restriction has the exponents $(n+u-1, n-s)$ with $n+u-1 > n+r$. Then $b_2(\mathcal{A})-d_1(\mathcal{A})d_2(\mathcal{A}) = n(n+r)-d_1(\mathcal{A})d_2(\mathcal{A}) > 0$ since $d_2(\mathcal{A}) > n + r$ by Lemma 2.10. Hence Theorem 2.2 implies that $\mathcal{A}$ is not free. $\square$

It is natural to ask whether the same statement as in Theorem 1.5 holds true for $s \leq 0$. The answers is affirmative as follows.

Proposition 4.4
In the same notation and condition as in Theorem 1.5, we assume that $-r \leq s \leq 0$. Then $\mathcal{A}$ is free if and only if $n_H \in \{n, n + r\}$ for some $H \in \mathcal{A}$.

Proof. The "if" part follows by Theorem 1.1 (3). Conversely, assume that $\mathcal{A}$ is free and $n_H \notin \{n, n + r\}$. Then Theorem 1.1 (1) shows that $n_H < n$ or $n_H > n + r$. Since $\mathcal{A}$ is free, Theorem 2.2 and Lemma 2.9 (2) imply that $n_H < n$. Let $\mathcal{A} \setminus \mathcal{B} = \{H_1, \ldots, H_{r+s+1}\}$. Put B_i and n_i in the same way as in Theorem 1.5 by Lemma 4.3. Then Theorem 1.1 and Corollary 1.2 show that $n - 1 \leq n_1 \leq n_{r+s+1} \leq n - 1$. However,

$$b_2(\mathcal{A}) = (n-1)(n-s)+(r+s+1)(n-1) = (n-1)(n+r+1) < n(n+r) = b_2(\mathcal{A}),$$

which is a contradiction. $\square$

Before the proof of Theorem 1.6, we need the following lemma.

Lemma 4.5
Let $\mathcal{A} \supset \mathcal{B}$ be the same arrangments as in Theorem 1.6. Let us order $\mathcal{A}\setminus\mathcal{B} = \{H_1, \ldots, H_f\}$ ($f := 2n+r-\alpha-\beta$) in such a way that $\mathcal{B}_0 := \mathcal{B}$, $\mathcal{B}_i := \mathcal{B}_{i-1} \cup \{H_i\}$ and $n_1 \le \cdots \le n_f$ for $n_i := |\mathcal{B}_{i-1} \cap H_i|$ by Lemma 4.3. Let a be the smallest integer satisfying $\alpha \le a$. Assume that $n_f \le n-1$, $n-1 < \beta$ and put $\chi(\mathcal{B}_i, t) = (t-\alpha_i)(t-\beta_i)$ with $|\alpha_i| \le |\beta_i|$ ($i = 1, \ldots, f$). Then α_i and β_i are both real numbers, and $\alpha_{i+1} \le \alpha_i \le \alpha \le \beta \le \beta_i \le \beta_{i+1}$ for any i. In particular, $n_i \le a$ for $i = 1, \ldots, f$.

Proof. Let us prove by induction on i. Since $\chi(\mathcal{B}, t) = (t-\alpha)(t-\beta)$, Theorem 1.1 (1) shows the case $i = 0$. Assume that the statement is true when $i \le k$. Since $n-1 < \beta \le \beta_k$, it holds that $n_{k+1} \le \alpha_k$ by Theorem 1.1 (2). Since

$$\chi(\mathcal{B}_{k+1}, t) = t^2 - (\alpha_k + \beta_k + 1)t + \alpha_k\beta_k + n_{k+1},$$

the roots of this polynomial are of the form

$$t_\pm = \frac{\alpha_k + \beta_k + 1 \pm \sqrt{(\alpha_k - \beta_k - 1)^2 + 4(\alpha_k - n_{k+1})}}{2}.$$

Since $\alpha_k \ge n_{k+1}$, it follows that $t_\pm \in \mathbb{R}$. Also, it is easy to see that $t_- \le \alpha_k$ and $\beta_k \le t_+$. Hence Theorem 1.1 (1) completes the proof. $\square$

Proof of Theorem 1.6. The "if" part follows from Theorem 1.1 (3). Assume that $\mathcal{A}$ is free and there are no $H \in \mathcal{A}$ such that $n_H \in \{n, n+r\}$. By Lemma 2.8 we may assume that $n_H \le n+r$. Hence Theorem 1.1 (1) shows that $n_H \le n-1$ for $H \in \mathcal{A}$. Let us order $\mathcal{A} \setminus \mathcal{B} = \{H_1, \ldots, H_f\}$ ($f := 2n+r-\alpha-\beta$) in such a way that $\mathcal{B}_0 := \mathcal{B}$, $\mathcal{B}_i := \mathcal{B}_{i-1} \cup \{H_i\}$ and $n_1 \le \cdots \le n_f$ for $n_i := |\mathcal{B}_{i-1} \cap H_i|$ by Lemma 4.3.

Case 1. Assume that $\alpha \le n-1 \le \beta$. If $\beta = n-1$, then

$$\begin{aligned} b_2(\mathcal{A}) &\le \alpha(n-1) + (n-1)(2n+r-n+1-\alpha) \\ &= (n-1)(n+r+1) < n(n+r) = b_2(\mathcal{A}), \end{aligned}$$

which is a contradiction. Hence we may assume that $n-1 < \beta$.

Let a, b be integers such that $\alpha \le a < \alpha+1$ and $\beta - 1 < b \le \beta$. Hence $\alpha + \beta = a + b$. Since $\alpha + \beta = |\mathcal{A}| \in \mathbb{Z}$, it holds that $a \le n-1 \le b$ and $\alpha\beta \le ab$. Since $n-1 < \beta$, we may apply Lemma 4.5 to obtain that $n_i \le a$. Hence

$$\begin{aligned} b_2(\mathcal{A}) &\le ab + a(2n+r-a-b) \\ &= a(2n+r-a) < n(n+r) = b_2(\mathcal{A}), \end{aligned}$$

which is a contradiction.

Case 2. Assume that $n-1<\alpha\le\beta<n$. Then $\alpha+\beta=2n-1$ and $\alpha\beta\le(n-\frac{1}{2})^2$. Hence

$$\begin{aligned} b_2(\mathcal{A}) &\le (n-\frac{1}{2})^2+(n-1)(2n+r-2n+1) \\ &= (n-1)(n+r+1)+\frac{1}{4}<n(n+r)=b_2(\mathcal{A}), \end{aligned}$$

which is a contradiction.

Case 3. Assume that $n-1<\alpha\le n,\ n\le\beta$. Let a and b be the same integers as in the Case 1. Hence $n\le b$ and $a=n$. Since $n_i\le n-1$ and $n\le\beta$, we may apply Lemma 4.5 to obtain that $n_i\le a$. Hence

$$\begin{aligned} b_2(\mathcal{A}) &\le nb+n(2n+r-n-b) \\ &= n(n+r)=b_2(\mathcal{A}). \end{aligned}$$

The equality holds only when $\alpha=n=n_1=\cdots=n_f$, which contradicts $n_f\le n-1$. $\square$

5 Applications related to the conjecture of Terao

In this section we study the relation between the conjecture of Terao and the results in the previous sections.

First, let us show the following proposition, which is a generalization of Theorem 1.5 in a special case.

Proposition 5.1
Let $\mathcal{A}$ be an affine line arrangement such that $\chi(\mathcal{A},t)=(t-n)(t-n-r)$ with $n\in\mathbb{Z}_{\ge0}$ and $r\in\mathbb{Z}_{\ge1}$. Assume that $\mathcal{A}$ contains a free subarrangement $\mathcal{B}$ with $\exp_0(\mathcal{B})=(n-2,n-2)$. Then $\mathcal{A}$ is free if and only if $n_H=n$ or $n+r$ for some $H\in\mathcal{A}$.

Proof. The "if" part follows by Theorem 1.1 (3). Assume that $\mathcal{A}$ is free and $n_H\notin\{n,n+r\}$. Then $n_H>n+r$ or $n-2\le n_H<n$ by Theorem 1.1 and Corollary 1.2. Also, $n_H>n+r$ implies the non-freeness of $\mathcal{A}$ by Theorem 2.2 and Lemma 2.9 (2). Hence we may assume that $n_H<n$.

Let $\{H_1,\ldots,H_{r+4}\}=\mathcal{A}\setminus\mathcal{B}$. Put $\mathcal{B}_0:=\mathcal{B}$, $\mathcal{B}_i:=\mathcal{B}_{i-1}\cup\{H_i\}$ $(i=1,\ldots,r+4)$. Then for $n_i:=|H_i\cap\mathcal{B}_{i-1}|$, we may assume that $n_1\le n_2\le\cdots\le n_{r+4}<n$ by Lemma 4.3. Also, Corollary 1.2 shows that $n-2\le n_1$. Let $u\ge1$ be the integer such that $n_i=n-2$ if $i\le u$ and $n_i=n-1$ if $i>u$.

Then an easy computation shows that $u = -r < 0$, which is a contradiction. $\square$

Theorem 5.1 has the following corollary.

Corollary 5.2
Let $\mathcal{A}$ be an affine line arrangement.
(1) If $\chi(\mathcal{A}, t) = (t-2)(t-2-r)$ with $r > 0$, then the freeness of $\mathcal{A}$ depends only on combinatorics.
(2) If $L(\mathcal{A})$ contains a point and $\chi(\mathcal{A}, t) = (t-3)(t-3-r)$ with $r > 0$, then the freeness of $\mathcal{A}$ depends only on combinatorics.

Proof. (1) Since $\mathcal{A}$ contains an empty arrangement with exponents $(0, 0)$, Corollary 5.1 completes the proof.
(2) Since $\mathcal{A}$ contains a Boolean arrangement with exponents $(1, 1)$, Corollary 5.1 completes the proof. $\square$

The following can be proved by the same way as in Theorem 5.1.

Proposition 5.3
Let $\mathcal{A}$ be an affine line arrangement such that $\chi(\mathcal{A}, t) = (t-n)(t-n-r)$ with $n, r \in \mathbb{Z}_{\geq 0}$.
(1) Assume that $r \geq 2$ and $\mathcal{A}$ contains a free subarrangement $\mathcal{B}$ with $\exp_0(\mathcal{B}) = (n-2, n-3)$. Then $\mathcal{A}$ is free if and only if $n_H = n$ or $n + r$ for some $H \in \mathcal{A}$.
(2) Assume that $r \geq 4$ and $\mathcal{A}$ contains a free subarrangement $\mathcal{B}$ with $\exp_0(\mathcal{B}) = (n-3, n-3)$. Then $\mathcal{A}$ is free if and only if $n_H = n$ or $n + r$ for some $H \in \mathcal{A}$.

On the conjecture of Terao, which asserts that the freeness of an arrangement $\mathcal{A}$ depends only on its combinatorics $L(\mathcal{A})$, we can give a few contribution by using these with Theorem 2.7. The conjecture of Terao for line arrangements in $\mathbb{C}^2$ is confirmed when $|\mathcal{A}| \leq 10$ by Wakefield-Yuzvinsky ([WY], Corollary 7.5), and $|\mathcal{A}| \leq 11$ by Faenzi-Vallès. ([FV], Theorem 5).

Now using the results in this article, first, we can show the following.

Corollary 5.4
Let $\mathcal{A}$ be an affine line arrangement in $\mathbb{C}^2$ such that $\chi(\mathcal{A}, t) = (t-n)(t-n-r)$ with $n, r \in \mathbb{Z}_{\geq 0}$. If $r \geq n-3$, then the freeness of $\mathcal{A}$ depends only on $L(\mathcal{A})$.

Proof. Let $(\mathcal{A}'', m)$ be the Ziegler restriction of $c\mathcal{A}$ onto $\{z = 0\}$. By Lemma 2.8, we may assume that $\mathcal{A}$ and $(\mathcal{A}'', m)$ are balanced. Put $\exp(\mathcal{A}'', m) = (d_1, d_2)$ with $d_1 \leq d_2$. By Theorem 2.7 (2), we know that the combinatorial invariant $h := |\mathcal{A}''| \geq r + 2$. When $h = r + 2$ or $r + 3$, the freeness of $\mathcal{A}$

is confirmed by Theorem 2.7 (2). Assume that $h \geq r+4 \geq n+1$. Then Theorem 1.1 (1) shows that $h \notin \{n+2, \ldots, n+r\}$, and Theorem 1.1 (3) shows that $\mathcal{A}$ is free when $h = n+1$ or $n+r+1$. Also, the non-freeness of $\mathcal{A}$ when $h > n+r+1$ is checked in [WY], or by applying Theorem 2.2 and Lemma 2.9 (2). $\square$

Using Corollary 5.4, in this article, we check the conjecture of Terao from a different point of view from [WY] and [FV]. Namely, we prove the conjecture under the restriction on the roots of characteristic polynomials, not on the number of lines.

Corollary 5.5
Let $\mathcal{A}$ be an affine line arrangement in $\mathbb{C}^2$ such that $\chi(\mathcal{A}, t) = (t-n)(t-n-r)$ with $n, r \in \mathbb{Z}_{\geq 0}$. If $\{n, n+r\} \cap \{0,1,2,3,4,5\} \neq \emptyset$, then the freeness of $\mathcal{A}$ depends only on $L(\mathcal{A})$.

Proof. If $\{n, n+r\} \cap \{0, 1\} \neq 0$, then the conjecture of Terao is easy to check. Assume that $n+r \in \{2,3,4,5\}$. Then [WY] and [FV] complete the proof. So we may assume that $n \in \{2,3,4,5\}$. Also, the case $r = 0$ can be verified by [WY] and [FV]. So assume that $r > 0$.

Assume that $n = 2$. Then Corollary 5.2 (1) completes the proof. Assume that $n = 3$. Then a point is contained in $L(\mathcal{A})$. Hence Corollary 5.2 (2) completes the proof.

Assume that $n = 4$. By Lemma 2.8, we may assume that $\mathcal{A}$ is balanced. Then Corollary 5.4 verifies the statement when $r \geq 1$. Hence it suffices to check when $\chi(\mathcal{A}, t) = (t-4)^2$, which is checked in [WY] and [FV].

Assume that $n = 5$. By Lemma 2.8, we may assume that $\mathcal{A}$ is balanced. Then Corollary 5.4 verifies the statement when $r \geq 2$. Hence it suffices to check when $\chi(\mathcal{A}, t) = (t-5)(t-6)$ or $(t-5)^2$, which is checked in [FV]. $\square$

6 The case over finite fields

In this section let us consider the case when $\mathbb{K}$ is a finite field $\mathbb{F}_q$. We give an another proof of Theorem 10 in [Yo2]. Also, we give a new sufficient condition for freeness which is a similar result to that in [Yo2]. Namely, in [Yo2], it is shown that an arrangement which has q as the root of the characteristic polynomial is free. Here we show that the same holds true when $q-1$ is a root.

In this section we use the following setup. Let $\mathbb{F}_q$ be a finite field of cardinality $q = p^n$ for a prime number p and $V = \mathbb{F}_q^2$. Recall that, for an affine arrangement $\mathcal{A}$ in V, it holds that

$$\chi(\mathcal{A}, q) = |V \setminus \cup_{H \in \mathcal{A}} H|.$$

See Theorem 2.69 in [OT] for example. Now consider a multiarrangement $(\mathcal{A}, m)$ in V. Put $\exp(\mathcal{A}, m) = (d_1, d_2)$ with $d_1 \le d_2$.

Proposition 6.1
Assume that $m(H) \le q$ for any $H \in \mathcal{A}$. Then
(1) the inequality $d_1 < q < d_2$ cannot occur.
(2) If $|m| \ge 2q$, then $d_1 = q$.
(3) If $|m| = 2q - 1$, then $d_2 = q$.

Proof. (1) Let θ_1, θ_2 be a basis for $D(\mathcal{A}, m)$ with $\deg \theta_i = d_i$. Assume that $d_1 < q < d_2$. Note that $\varphi := x^q \partial_x + y^q \partial_y \in D(\mathcal{A}, m)$, which is of degree q. Hence $\varphi = f\theta_1$ for some polynomial f. Since φ has no divisors in $\mathrm{Der}(S')$, this is a contradiction.
(2) By (1) and $|\mathcal{A}| = d_1 + d_2 \ge 2q$, we know that $d_2 \ge d_1 \ge q$. Since $\varphi \in D(\mathcal{A}, m)$, we know that $d_1 \le q$, which completes the proof.
(3) By assumption, $d_2 \ge q$. If $d_2 > q$, then $d_1 < q < d_2$, which is a contradiction. □

Corollary 6.2 ([Yo2], Theorem 10)
Let $\mathcal{A}$ be an affine line arrangement in V.
(1) If $\chi(\mathcal{A}, q) = 0$, then $\mathcal{A}$ is free.
(2) If $|\mathcal{A}| \ge 2q - 1$ and $\mathcal{A}$ is free, then $\chi(\mathcal{A}, q) = 0$.

Proof. Let $(\mathcal{A}'', m)$ be the Ziegler restriction of $c\mathcal{A}$ onto $z = 0$. Put $\exp(\mathcal{A}'', m) = (d_1, d_2)$ with $d_1 \le d_2$. Note that $d_1 + d_2 = |\mathcal{A}|$. Also, note that we may apply Proposition 6.1 since the base field is $\mathbb{F}_q$.

(1) Let $\chi(\mathcal{A}, t) = (t - q)(t - r)$. Note that $q + r = d_1 + d_2 = |\mathcal{A}| = |m|$. First assume that $r \le q$. Then Theorem 2.2 implies that $qr \ge d_1 d_2$. Hence $d_1 \le r \le q \le d_2$. By Proposition 6.1 (1), we know that $q = d_2$. Hence $\mathcal{A}$ is free by Theorem 2.2.

Second assume that $r > q$. Then again the inequalities $d_1 \le q < r \le d_2$ and Proposition 6.1 (1) show that $d_1 = q$, which implies the freeness.

(2) Since $|m| = |\mathcal{A}| \ge 2q - 1$, Proposition 6.1 (2) and (3) imply that $d_1 = q$ or $d_2 = q$. Then the freeness of $\mathcal{A}$, Theorems 2.4 and 2.5 complete the proof. □

By applying Theorem 1.1, we can prove the following new result on arrangements in $\mathbb{F}_q^2$.

Theorem 6.3
Let $\mathcal{A}$ be an affine arrangement in $V = \mathbb{F}_q^2$. If $\chi(\mathcal{A}, q-1) = 0$, then $\mathcal{A}$ is free.

Proof. Put $\chi(\mathcal{A}, t) = (t - q + 1)(t - q + r)$ with $r \in \mathbb{Z}$. Since $\chi(\mathcal{A}, q) = r = |V \setminus \cup_{H \in \mathcal{A}} H| \geq 0$, we know that $r \in \mathbb{Z}_{\geq 0}$, and $\mathcal{A}$ is free if $r = 0$ by Corollary 6.2. Assume that $r \geq 1$. Since $\chi(\mathcal{A}, 0) \geq 0$, it holds that $\chi(\mathcal{A}, q) = r \leq q$. Let $V \setminus \cup_{H \in \mathcal{A}} H = \{p_1, \ldots, p_r\}$ and we may assume that p_1 is the origin. Then there are $(q + 1)$-lines containing p_1 and not belonging to $\mathcal{A}$. Hence there is at least one line $L \notin \mathcal{A}$ such that $p_1 \in L$ and $p_i \notin L$ for $i = 2, \ldots, r$. Then $|\mathcal{A} \cap L| = q - 1$. Hence Theorem 1.1 (3) shows that $\mathcal{A}$ is free. $\square$

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